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Ehlers Transformations and String Effective Action

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Abstract

We explicitly obtain the generalization of the Ehlers transformation for stationary axisymmetric Einstein equations to string theory. This is accomplished by finding the twist potential corresponding to the moduli fields in the effective two dimensional theory. Twist potential and symmetric moduli are shown to transform under an $O(d,d)$ which is a manifest symmetry of the equations of motion. The non-trivial action of this $O(d,d)$ is given by the Ehlers transformation and belongs to the set $\frac{O(d)\times O(d)}{O(d)}$.

Symmetries of stationary axisymmetric Einstein equations have been extensively studied in past decades[1, 2]. An interesting aspect of this system is the existence of an infinite dimensional symmetry, known as the Geroch group[3]. It has also been realized that the infinite symmetry structure in this case renders the model integrable[4]. This system describes an Ernst Sigma model[1, 2] which has been studied a great deal including for the supergravity models in two dimensions. The existence of an infinite number of conserved currents and their algebraic structure have also been analyzed[4, 5]. These symmetry transformations have also been used for generating solutions of Einstein equations such as Tomimatsu-Sato class of metrics[6].

Some aspects of the Ernst sigma model and the infinite dimensional symmetries have been analyzed recently in the context of string theory[7, 8, 9, 10]. In particular, the duality symmetries of the four dimesnsional string theory have been shown to be a subgroup of the infinite dimensional String Geroch group in the presence of two commuting isometries.

In this paper we further study the symmetries of string theory and obtain the analogue of the Ehlers transformation. We also show that, as in the case of Einstein equations, this Ehlers transformation is the one responsible for the existence of an infinite number of conserved currents in the theory and for generating new solutions. We in fact show the existence of a new $O(d,d)$ group of symmetry transformations of which Ehlers is a subset. The present approach has the advantage that we can write down the finite transformations of the fields directly.

An interesting scheme for finding out new solutions of stationary axisymmetric Einstein equations through the underlying symmetry transformations was developed in a series of papers by Kinnersley et al[11, 12]. They showed the existence of two sets of $SL(2,R)$ symmetries in this theory denoted as the groups $\mathcal{G}$ and $\mathcal{H}$. The action of two of the generators of $\mathcal{G}$ and $\mathcal{H}$, which correspond to translation and scaling of the

corresponding variables, coincide. However, the third generators of these groups do not commute with the dualization, *viz.* the Kramer-Neugebauer(KN) mapping[13]. This fact gives rise to an infinite set of conserved currents.

In this paper we generalize the approach of [11, 12] to string theory. We find out a twist potential corresponding to the antisymmetric moduli field. It is also shown that the equations of motion, written in terms of the twist potential, are similar to the original ones and are in fact related by a generalization of the KN-mapping. We show the existence of two sets of $O(d,d)$ symmetries. The first one acts on the usual moduli of string theory. The second $O(d,d)$ acts on the moduli involving the twist potentials. These are the analogues of the groups $\mathcal{G}$ and $\mathcal{H}$ in the present case. Now the set of generators corresponding to the constant shift of the antisymmetric tensor and the constant coordinate transformation in $\mathcal{G}$ and $\mathcal{H}$ coincide. The action of the remaining one in $\mathcal{H}$ is identified as the generalized Ehlers transformation. We also write down the action of the generalized Ehlers transformation on the original field variables.

We now begin our study of the bosonic part of the four dimensional heterotic string effective action for the case when the $E_8 \times E_8$ gauge field backgrounds are set to zero. In the presence of two commuting isometries, the equations of motion of this theory coincide with those following from the two dimensional string effective action. Therefore, we concentrate on the 2D action and the corresponding equations of motion. We further restrict to the case when the two dimensional gauge fields have vanishing background. This is consistent with the fact that gauge fields have no physical degrees of freedom in two dimensions. The action can then be written as

$$S = \int d^2x \sqrt{g} e^{-2\phi} \left[R + 4 g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + \frac{1}{8} g^{\mu\nu} Tr(\partial_\mu M^{-1} \partial_\nu M) \right], \tag{1}$$

where the matrix M, representing the moduli G and B, is parametrized as

$$M = \begin{pmatrix} G^{-1} & -G^{-1}B \\ BG^{-1} & G - BG^{-1}B \end{pmatrix}. \tag{2}$$

G and B are respectively $d \times d$ symmetric and antisymmetric matrix-valued scalar fields. For the heterotic string $d = 8$, but we keep it arbitrary in the present discussion. The equations of motion for the above action can be written as

$$\partial_\mu(\sqrt{g}g^{\mu\nu}e^{-2\phi}M^{-1}\partial_\nu M) = 0 \tag{3}$$

$$\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu e^{-2\phi}) = 0 \tag{4}$$

$$R^{(2)}_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi + \frac{1}{8}Tr(\partial_\mu M^{-1}\partial_\nu M) = 0. \tag{5}$$

The form of these equations are similar to the ones for the Einstein equations in [11] and the matrix M plays the role of the 2×2 metric in the case of Einstein equations. The equations (3) and (4) can be rewritten in the conformal gauge, with an identification $\rho = e^{-2\phi}$, as

$$\partial^\mu(\rho M^{-1}\partial_\mu M) = 0, \tag{6}$$

$$\partial^\mu\partial_\mu\rho = 0. \tag{7}$$

The third equation (5) turns out to be an integrable one for the conformal factor and takes the following form in the conformal gauge $g_{\mu\nu} = e^{2\Gamma}\delta_{\mu\nu}$:

$$\begin{aligned} -\delta_{\mu\nu}\partial^\sigma\partial_\sigma\Gamma + 2(\partial_\mu\partial_\nu\phi &- \partial_\mu\phi\partial_\nu\Gamma - \partial_\nu\phi\partial_\mu\Gamma + \delta_{\mu\nu}\partial^\sigma\Gamma\partial_\sigma\phi) \\ &+ \frac{1}{8}Tr(\partial_\mu M^{-1}\partial_\nu M) = 0. \end{aligned} \tag{8}$$

Determination of this conformal factor is important for the generation of new solutions. Here we have chosen the two dimensional metric to be Euclidean. This corresponds to the contraction of the Levi-Civita tensor: $\epsilon^{\mu\alpha}\epsilon_{\mu\beta} = \delta^\alpha_\beta$. All the results go through for the Minkowski space as well with few changes in signs. We now concentrate on equations (6), (7) and will come back to (8) later.

Equation (6) and (7) are the same as that for the Ernst sigma model[1, 5, 7]. It is also clear that these equations are invariant under an $O(d,d)$ group of symmetry transformations:

$$M \to \tilde{M} = \Omega M \Omega^T, \tag{9}$$

which includes the T-duality symmetry of the string theory as its discrete subgroup $O(d,d;Z)$ and has been used for generating solutions. The non-trivial action of T-duality is represented by a group $\frac{O(d;Z)\times O(d;Z)}{O(d;Z)}$[14]. The T-duality is also conjectured to be a symmetry, not only of the string effective action, but of the full string theory[15] too.

The Ehlers transformation on the moduli is now determined on the lines of [11] by obtaining the twist potentials. For this we observe that equations (6) are total divergence conditions and when written in terms of G and B, take the form

$$\partial^\mu(\rho G^{-1}\partial_\mu B G^{-1}) = 0 \tag{10}$$

$$\partial^\mu[\rho(G\partial_\mu G^{-1} + BG^{-1}\partial_\mu BG^{-1})] = 0 \tag{11}$$

$$\partial^\mu[\rho(\partial_\mu B + BG^{-1}\partial_\mu BG^{-1}B + G\partial_\mu G^{-1}B - BG^{-1}\partial_\mu G)] = 0. \tag{12}$$

One can also show that there are only two algebraically independent equations among (10)–(12), which is due to the fact that the M-equation is essentially a combination of the two equations for G and B. One can now define a set of three potentials by the following relations:

$$\rho G^{-1}\partial_\mu BG^{-1} = -\epsilon_{\mu\nu}\partial^\nu\psi_1, \tag{13}$$

$$\rho(G\partial_\mu G^{-1} + BG^{-1}\partial_\mu BG^{-1}) = -\epsilon_{\mu\nu}\partial^\nu\psi_2, \tag{14}$$

$$\rho(\partial_\mu B + BG^{-1}\partial_\mu BG^{-1}B + G\partial_\mu G^{-1}B - BG^{-1}\partial_\mu G) = -\epsilon_{\mu\nu}\partial^\nu\psi_3, \tag{15}$$

such that the equations (10)–(12) are the integrability conditions for the ψ_i's. Equations (13)–(15) are matrix generalization of the dualization relations in [11] and the

twist potential $\psi \equiv \psi_1$ is the dual field corresponding to the antisymmetric moduli B. One can use G, ψ and ρ to rewrite the equations (10)–(12). For example, an equation similar to (10) can be obtained by inverting (13) as,

$$\partial^\mu(\frac{G}{\rho}\partial_\mu\psi G) = 0. \tag{16}$$

Equation (11), when expressed in terms of ψ, takes the form

$$\partial^\mu(\rho G\partial_\mu G^{-1}) + \frac{1}{\rho}G\partial_\mu\psi G\partial^\mu\psi = 0. \tag{17}$$

Then using (16), (17) and (7), the last equation, (12), is satisfied identically.

The equations of motion in G, ψ variables, (16) and (17), can also be expressed as

$$\partial^\mu(\rho M'^{-1}\partial_\mu M') = 0, \tag{18}$$

where

$$M' = \begin{pmatrix} \frac{G}{\rho} & \frac{G}{\rho}\psi \\ -\psi\frac{G}{\rho} & -\rho G^{-1} - \psi\frac{G}{\rho}\psi \end{pmatrix}. \tag{19}$$

Equation (18) is one of the main results of this paper as it leads to new symmetries of this theory. The equations in terms of M' are related to those in terms of M through a KN-mapping of the form $G \to \rho G^{-1}, \quad B \to i\psi$.

One can also introduce a set of three potentials for M' as

$$\begin{aligned} \epsilon_{\mu\nu}\partial^\nu\Psi_1 &= -\frac{1}{\rho}G\partial_\mu\psi G, && (20) \\ \epsilon_{\mu\nu}\partial^\nu\Psi_2 &= -G\partial_\mu(\rho G^{-1}) - \frac{G}{\rho}\partial_\mu\psi G\psi, && (21) \\ \epsilon_{\mu\nu}\partial^\nu\Psi_3 = -\rho^2 G^{-1}\partial_\mu(\frac{G}{\rho})\psi &- \rho\partial_\mu\psi + \psi G\partial_\mu(\rho G^{-1}) + \psi G(\partial_\mu\psi)\frac{G}{\rho}\psi, && (22) \end{aligned}$$

whose integrability conditions give back the equations of motions for G and ψ. Comapring (20) with (13) we get $\Psi_1 = B$. Equation (5) can also be rewritten in

terms of G and ψ. For this we first rewrite (8) using (17) as

$$\begin{aligned}
-\delta_{\mu\nu}\partial^\sigma\partial_\sigma\Gamma &+ 2\left(\partial_\mu\partial_\nu\phi - \partial_\mu\phi\partial_\nu\Gamma - \partial_\nu\phi\partial_\mu\Gamma + \delta_{\mu\nu}\partial^\sigma\Gamma\partial_\sigma\phi\right) \\
&+ \frac{1}{4\rho^2}\delta_{\mu\nu}Tr\left(\partial^\sigma\psi G\partial_\sigma\psi G\right) \\
&- \frac{1}{4}Tr\left(G^{-1}\partial_\mu G G^{-1}\partial_\nu G + \frac{1}{\rho^2}\partial_\mu\psi G\partial_\nu\psi G\right) = 0.
\end{aligned} \tag{23}$$

It can be shown that under a transformation

$$\Gamma' = \Gamma - \frac{1}{4}\ln\det G - \frac{d}{4}\phi, \tag{24}$$

equation (23) goes over to

$$\begin{aligned}
-\delta_{\mu\nu}\partial^\sigma\partial_\sigma\Gamma' &+ 2\left[\partial_\mu\partial_\nu\phi - \partial_\mu\phi\partial_\nu\Gamma' - \partial_\nu\phi\partial_\mu\Gamma' + \delta_{\mu\nu}\partial^\sigma\Gamma'\partial_\sigma\phi\right] \\
&- \frac{1}{2}\left[\partial_\mu\phi\partial_\nu(\ln\det G) + \partial_\nu\phi\partial_\mu(\ln\det G) + 2d\partial_\mu\phi\partial_\nu\phi\right] \\
&- \frac{1}{4}Tr\left(G^{-1}\partial_\mu G G^{-1}\partial_\nu G + \frac{1}{\rho^2}\partial_\mu\psi G\partial_\nu\psi G\right) = 0.
\end{aligned} \tag{25}$$

which can be further rewritten as

$$R_{\mu\nu} + 2\nabla_\mu\nabla_\nu\phi + \frac{1}{8}Tr(\partial_\mu M'^{-1}\partial_\nu M') = 0 \tag{26}$$

with a new conformal factor Γ'. (7), (18) and (26) are the complete set of equations of motion rewritten in terms of G and ψ.

We now study the symmetries of equations (18). Like the M equations these are also manifestly invariant under a transformation

$$M' \to \tilde{M'} = \Omega' M' \Omega'^T, \tag{27}$$

However, since ψ is an antisymmetric matrix due to the dualization relation (13), M' satisfies

$$M'LM' = -L, \quad \text{where} \quad L = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \tag{28}$$

and I is the d dimensional identity matrix. The condition (28) is maintained under (27) provided Ω' satisfies the relation

$$\Omega'^T L \Omega' = L. \tag{29}$$

As a result, the matrices Ω' belong to the group $O(d,d)$.

We have therefore shown that a new type of $O(d,d)$ symmetry transformation can be used for generating solutions of the string effective action. Like the $o(d,d)$ algebra of the symmetry transformations in equations (6), this is also expected to be the part of an affine $\hat{o}(d,d)$. We have been able to explicitly identify the action of the new $O(d,d)$ by making use of dualization.

We now show that only a part of the new $O(d,d)$ has non-trivial action to give infinite number of conservation laws and generate new solutions. For this, we classify the independent $O(d,d)$ transformations into three sets:

$$\Omega_1 = \begin{pmatrix} I & \gamma \\ 0 & I \end{pmatrix} \quad \Omega_2 = \begin{pmatrix} I & 0 \\ \alpha & I \end{pmatrix} \quad \Omega_3 = \begin{pmatrix} A^{-1T} & 0 \\ 0 & A \end{pmatrix} \tag{30}$$

where γ and α are real antisymmetric matrices and A is an arbitrary nonsingular real $d \times d$ matrix.

A different realization of $O(d,d)$ matrices, to study T-duality symmetries, was given in [14], where the non-trivial part of $O(d,d)$ was identified as $\frac{O(d)\times O(d)}{O(d)}$. We have chosen the above representaton in order to make connection with the Ehlers transformation of General Relativity[11]. In [14] it was shown that the matrices Ω_2 and Ω_3 together parametrize the generators of $O(d,d)$ which are outside a set $\frac{O(d)\times O(d)}{O(d)}$. As a result, one can identify Ω_1 as one which parametrizes $\frac{O(d)\times O(d)}{O(d)}$. The $O(d,d)$ transformations of G and ψ under the Ω_i's can be obtained from the condition $\tilde{M'}_i = \Omega_i M' \Omega_i^T$. The transformation under Ω_1 is written as

$$\text{(i)} \quad \frac{G}{\rho} \quad \rightarrow \quad \frac{G}{\rho} - (\frac{G}{\rho}\psi\gamma + \gamma\psi\frac{G}{\rho}) + \gamma(\rho G^{-1} + \psi\frac{G}{\rho}\psi)\gamma$$

$$-\frac{\psi G}{\rho} \quad \rightarrow \quad -\frac{\psi G}{\rho} + (\rho G^{-1} + \psi\frac{G}{\rho}\psi)\gamma \tag{31}$$

leaving ρ, as well as $(\rho G^{-1} + \psi\frac{G}{\rho}\psi)$, invariant. The antisymmetry of ψ is ensured as the condition (28) is preserved under these transformations. Under Ω_2 and Ω_3 we have, respectively

$$\text{(ii)} \quad \frac{G}{\rho} \rightarrow \frac{G}{\rho}, \qquad \psi \rightarrow \psi - \alpha \tag{32}$$

$$\text{(iii)} \quad \frac{G}{\rho} \rightarrow {A^{-1}}^T\frac{G}{\rho}A^{-1}, \qquad \psi \rightarrow A\psi A^T. \tag{33}$$

The action of Ω's on B-fields is obtained from a relation $\partial_\mu \tilde{\chi}_i = {\Omega_i^{-1}}^T \partial_\mu \chi \Omega_i^T$, where

$$\chi = \begin{pmatrix} -\Psi_2^T & \Psi_3 \\ \Psi_1 & \Psi_2 \end{pmatrix}. \tag{34}$$

Consequently $B \equiv \Psi_1$ has the following transformations under Ω_i's.

$$\text{(i)} \quad B \rightarrow \tilde{B} = B - \Psi_2\gamma - \gamma\Psi_2^T - \gamma\Psi_3\gamma, \tag{35}$$

$$\text{(ii)} \quad B \rightarrow B \qquad \text{and} \qquad \text{(iii)} \quad B \rightarrow {A^{-1}}^T B A^{-1}. \tag{36}$$

From (32) and (36) we see that Ω_2 simply corresponds to a constant shift in B and ψ. The action of Ω_3 is same as a constant coordinate transformation. These, together with the results of [14], imply that the corresponding parts of $O(d,d)$ acting on M and M' in fact coincide. However, Ω_1 acts very differently on B and on ψ. Its non-local action on B originates from the fact that the potentials ψ_i are related to Ψ_i through integrations.

The infinite set of conserved currents are now obtained by applying the transformations Ω_i's on the dualization equation (13). It is first observed that the action of Ω_2 and Ω_3 leaves this equation invariant and therefore does not lead to any new conservation law. The transformation by Ω_1 on the other hand implies

$$\left[\frac{G}{\rho} \quad - \quad \left(\frac{G}{\rho}\psi\gamma + \gamma\psi\frac{G}{\rho}\right) + \gamma\left(\rho G^{-1} + \psi\frac{G}{\rho}\psi\right)\gamma\right]^{-1} \partial^\mu \left[B - \Psi_2\gamma - \right.$$

$$
\begin{aligned}
\gamma\Psi_2^T \quad - \quad & \gamma\Psi_3\gamma]\left[\frac{G}{\rho}-\left(\frac{G}{\rho}\psi\gamma+\gamma\psi\frac{G}{\rho}\right)+\gamma\left(\rho G^{-1}+\psi\frac{G}{\rho}\psi\right)\gamma\right]^{-1} \\
= \quad & -\epsilon^{\mu\nu}\rho\partial_\nu\left(\left[\psi\frac{G}{\rho}-\left(\rho G^{-1}+\psi\frac{G}{\rho}\psi\right)\gamma\right]\left[\frac{G}{\rho}-\left(\frac{G}{\rho}\psi\gamma\ +\right.\right.\right. \\
& \left.\left.\left.\gamma\,\psi\frac{G}{\rho}\right)+\gamma\left(\rho G^{-1}+\psi\frac{G}{\rho}\psi\right)\gamma\right]^{-1}\right). \qquad (37)
\end{aligned}
$$

The infinite set of conserved currents for this theory is now obtained by expanding both sides of this equation in powers of γ. For example, the zeroth power gives back the dualization equation (13) and the conservation equation (10). From the next order we get, after some calculations,

$$
\epsilon^{\mu\nu}\partial_\nu G=\frac{G}{\rho}(\partial^\mu\Psi_2-\partial^\mu B\psi), \qquad (38)
$$

which leads to another conservation equation:

$$
\partial_\mu\left[\frac{G}{\rho}(\partial^\mu\Psi_2-\partial^\mu B\psi)\right]=0. \qquad (39)
$$

as well as its transpose.

To conclude, in this paper we have obtained the explicit action of a new $O(d,d)$ transformation in the string effective action. We have identified the non-trivial part of this $O(d,d)$ as a generalized Ehlers transformation. We have also shown that the generalized Ehlers belongs to a set of $\frac{O(d)\times O(d)}{O(d)}$ transformations which is similar to the non-trivial part of the T-duality. Our results indicate that only a subset of the infinite dimensional symmetries represented by $\hat{o}(d,d)$ algebra of these theories has a role in generating new solutions.

We can also introduce an *Ernst* potential, $E=\rho G^{-1}-i\psi$, and combine the equations (16) and (17) for G and ψ into a single equation

$$
\partial^\mu\left(\rho\partial_\mu E\right)=2\rho\partial^\mu E\left(E+E^T\right)^{-1}\partial_\mu E. \qquad (40)
$$

This is a matrix generalization of a similar equation for Einstein gravity[1]. Hopefully this will lead to an *Ernst formulation* of the present problem.

As mentioned earlier, the results of this paper are also valid for the choice of the two dimensional metric being Minkowski. In this case the sign of the first term in the (22) element of M' in equation (19) is positive. However, the covariant forms of all the original as well as the dual equations of motion remain unchanged.

It will also be interesting to apply the Ehlers transformation obtained in this paper to various physical situations alone or together with T-duality and extend the results to heterotic string theories for non-zero $E_8 \times E_8$ gauge backgrounds. In this paper we have related the equations of motion of the original and dual fields. Possibly one can also write down the action and the equations of motion where both B and ψ appear on an equal footing. Such actions for the world-sheet theories as well as for four dimensional theories have already been written[15]. Some of these will be reported in future.

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